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Dynamics Model for a Multi-Tethered Space-Based Interferometer

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A mathematical dynamics model is established for a space-based interferometer concept comprised of multiple collector elements connected to a common central body by tethers of variable length. The system as a whole is to rotate in a coordinated fashion while the collectors simultaneously execute linear extension and retraction motions. The system is taken to consist of a single rigid body and N point mass collector elements connected to the rigid body by massless tethers. The equations of motion for the system are derived from fundamental Newton-Euler momentum principles. Tether constitutive behavior is discussed. Prescribed motion of any system degrees of freedom is treated, and expressions for the corresponding constraint force/torque components are established.

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Dynamics Model for a Multi-Tethered Space-Based Interferometer

Introduction

This report presents the derivation of the equations of motion for a simplified model of a novel space-based interferometer[1]. The instrument, intended for operation at the L2 Lagrange point, would be comprised of multiple collector elements carried by individual spacecraft whose motion would be precisely coordinated. The concept entails a symmetrical radial arrangement of the respective spacecraft connected to a common central vehicle by low mass tethers. During observations the system as a whole would rotate in a common plane while the collector elements simultaneously moved radially to scan the subtended circular area. Figure-1 illustrates one candidate configuration.

This report establishes a basic mathematical model of the conceptual system, develops the unabridged equations of motion, discusses tether constitutive behavior, and includes treatment of prescribed motion of designated degrees of freedom.

The mechanical system considered in this report is comprised of a single rigid body and N point masses. The point masses are taken to be connected to the rigid body at distinct points by idealized tethers. The tethers are treated as massless tensile members of variable length, capable of exerting force only along the straight-line connecting the respective endmasses with the rigid body attachment points. The tethers do not support compression nor any components of shear force or bending moment. The formulation is sufficiently general to allow consideration of both extensible and inextensible tether models.

For the development of the equations of motion the degrees of freedom of the system are left unrestricted, i.e. there are no small magnitude limitations. The rigid body can translate and rotate in three dimensions and the tethered endmasses are each provided three translational degrees of freedom. The system components are all subject to external forces while the rigid body is subject to external torque as well. The motion equations are derived from fundamental Newton-Euler momentum principles. The development of explicit expressions for gravitational and other environmental forces are not included here. To allow for the determination of forces and torques necessary to accomplish specified motions in any or all the degrees of freedom, kinematic constraints are imposed on the system. The altered form of the dynamics equations and the expressions for the constraint forces are developed. The equations governing the motion of both the constrained

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and unconstrained systems are brought to complete closed sets suitable for numerical implementation.

Idealization of the tethers as massless affords the model substantial analytical simplicity and numerical efficiency but at the cost of limiting the dynamic fidelity. Without distributed mass there is no way to account for transverse deflections or vibrations of the tether beyond fundamental pendular motions (present under rotation). For an extensible tether with a linear elastic or linear viscoelastic constitutive law the fundamental longitudinal vibration mode can be captured but all such higher modes are forfeited.

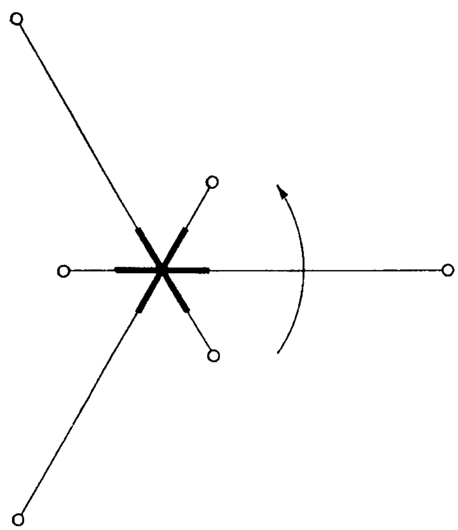

Figure-1. Concept Tethered Interferometer.

System Definition

The system consists of a rigid body R and N point masses P_i $(i=1,2,...,N)$. Particles P_i connect to points A_i, fixed to R, by massless tether members. The tethers are only capable of applying tensile force between their respective end points A_i and P_i. Full three dimensional motion is assumed, hence the system possesses 6+3N degrees of freedom. Reference point 0 is fixed to R at its mass center and establishes an origin for the R fixed reference frame, $\mathcal{F}_R$. Reference point I is an inertially fixed point, and is taken as the origin for an inertial reference frame, denoted $\mathcal{F}_I$. The system geometry as well as the following position vectors are shown in Figure-2.

$\vec{R}_0$ - position vector from point-I to point-0.

$\vec{a}_i$ - position vector from point-0 to point-A_i.

$\vec{p}_i$ - position vector from point-A_i to P_i.

$\vec{r}$ - position vector from point-0 to a material point of R.

The system, $(R + P_1 + P_2 + \ldots + P_N)$, is subject to the external forces and torques defined below.

$\vec{F}_i$ - resultant external force acting on P_i.

$\vec{F}_R$ - resultant external force acting on R.

$\vec{G}_R$ - resultant moment about point-0 of external forces acting on R.

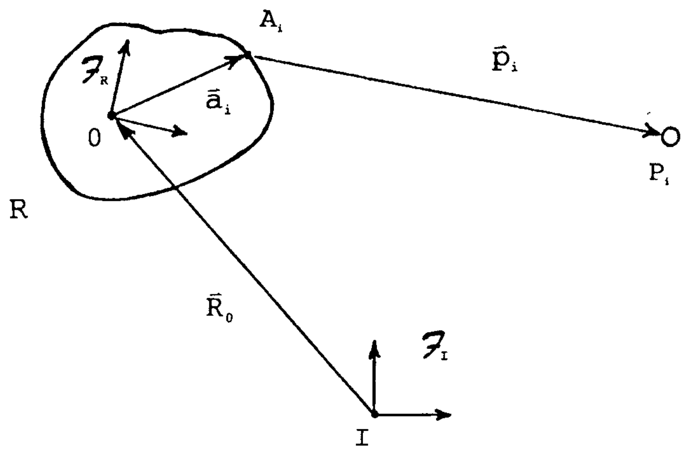

Figure-2. System Vector Geometry.

Let m_R denote the mass of R, and m_i be the mass of P_i. The total mass of the system is

$$m_t = m_R + \sum_{i=1}^{N} m_i$$

The position of the system mass center with respect to point-0 is given by

$$\vec{r}_c = \frac{1}{m_t} \sum_{i=1}^{N} m_i \left(\vec{a}_i + \vec{p}_i\right) \tag{1}$$

System Momenta

Throughout the following development the notation; $\dot{\vec{v}}$ designates the time derivative of vector quantity $\vec{v}$ with respect to the inertial frame $\mathcal{F}_I$. Similarly, the notation $\mathring{\vec{v}}$ denotes time differentiation of the vector with respect to the R-fixed frame, $\mathcal{F}_R$. The angular velocity of $\mathcal{F}_R$ with respect to $\mathcal{F}_I$ is $\vec{\omega}$.

The linear momentum of P_i is

$$\vec{L}_i = m_i\left(\dot{\vec{R}}_0 + \dot{\vec{a}}_i + \dot{\vec{p}}_i\right)$$

Let the absolute velocity of point-0 be denoted $\vec{v}_0$, then

$$\vec{L}_i = m_i\left[\vec{v}_0 - \left(\vec{a}_i+\vec{p}_i\right)\times\vec{\omega} + \mathring{\vec{p}}_i\right] \tag{2}$$

The total linear momentum of the system is

$$\vec{L} = m_R\vec{v}_0 + \sum_{i=1}^{N}\vec{L}_i$$

or

$$\vec{L} = m_t\vec{v}_0 - m_t\vec{r}_c\times\vec{\omega} + \sum_{i=1}^{N} m_i\mathring{\vec{p}}_i \tag{3}$$

The absolute angular momentum[2] of the system about point-0 is

$$\vec{H}_0 = \int_R \vec{r}\times(\vec{v}_0 + \vec{\omega}\times\vec{r})\,dm + \sum_{i=1}^{N} m_i\left(\vec{a}_i+\vec{p}_i\right)\times\left(\vec{v}_0 + \vec{\omega}\times\left(\vec{a}_i+\vec{p}_i\right) + \mathring{\vec{p}}_i\right)$$

Expanding, simplifying, and collecting terms yields

$$\vec{H}_0 = \left(\vec{I}_R + \sum_{i=1}^{N}\vec{J}_i\right)\cdot\vec{\omega} + m_t\vec{r}_c\times\vec{v}_0 + \sum_{i=1}^{N} m_i\left(\vec{a}_i+\vec{p}_i\right)\times\mathring{\vec{p}}_i \tag{4}$$

where the following inertia dyadics have been introduced:

$\vec{I}_R$ - inertia dyadic of R with respect to point-0.

$\vec{J}_i$ - inertia dyadic of P_i with respect to point-0.

$$\vec{J}_i = m_i\left[\left(\vec{a}_i+\vec{p}_i\right)\cdot\left(\vec{a}_i+\vec{p}_i\right)\vec{1} - \left(\vec{a}_i+\vec{p}_i\right)\left(\vec{a}_i+\vec{p}_i\right)\right] \tag{5}$$

where $\vec{1}$ is the unit dyadic. Also of interest is the absolute angular momentum of the system about the system centroid

$$\vec{H}_c = \int_R (\vec{r} - \vec{r}_c)\times(\vec{v}_0 + \vec{\omega}\times\vec{r})\,dm + \sum_{i=1}^{N} m_i\left(\vec{a}_i+\vec{p}_i-\vec{r}_c\right)\times\left(\vec{v}_0 + \dot{\vec{a}}_i + \dot{\vec{p}}_i\right)$$

or

$$\vec{H}_c = \left[\vec{I}_R + \sum_{i=1}^{N} \vec{J}_i - m_t\left(r_c^2\, \vec{1} - \vec{r}_c\, \vec{r}_c\right)\right] \cdot \vec{\omega} + \sum_{i=1}^{N} m_i\left(\vec{a}_i + \vec{p}_i - \vec{r}_c\right) \times \mathring{\vec{p}}_i \qquad (6)$$

Dynamic Equilibrium Equations

Let $\vec{T}_i$ be the force on P_i from the tether connecting it to R. Newton's second law for P_i requires

$$\dot{\vec{L}}_i = \vec{F}_i + \vec{T}_i \qquad (7)$$

Considering the system as a whole, Newton's second law requires

$$\dot{\vec{L}} = \vec{F}_R + \sum_{i=1}^{N} \vec{F}_i \qquad (8)$$

The law of moment of momentum[2] supplies an additional vector equilibrium equation. When written with respect to point-0, that law requires that the system satisfy

$$\dot{\vec{H}}_0 + \vec{v}_0 \times \vec{L} = \vec{G}_R + \sum_{i=1}^{N} m_i\left(\vec{a}_i + \vec{p}_i\right) \times \vec{F}_i \qquad (9)$$

Equations (7)-(9) constitute 2+N independent vector-dyadic dynamic equilibrium equations for the system.

Scalar Equations

It is presumed that each tether has a preferred nominal direction relative to R. A reference frame $\mathcal{F}_i^0$ fixed to R, with origin at A_i, is established for each tether with its x-axis in the desired tether reference direction. The other two axes of $\mathcal{F}_i^0$ are oriented for convenience. The orientation of $\mathcal{F}_i^0$ with respect to $\mathcal{F}_R$ is defined in terms of a constant direction cosine matrix, $[C_{Ri}^0]$, which transforms the respective unit basis vectors according to

$$\begin{pmatrix} \hat{i}_R \\ \hat{j}_R \\ \hat{k}_R \end{pmatrix} = [C_{Ri}^0] \begin{pmatrix} \hat{i}_i^0 \\ \hat{j}_i^0 \\ \hat{k}_i^0 \end{pmatrix}$$

It proves convenient to define another reference frame $\mathcal{F}_i$, which moves such that its x-axis is always in the direction of the line

segment from A_i to P_i. The orientation of $\mathcal{F}_i$ with respect to $\mathcal{F}_i^0$ is described in terms of a sequence of two single axis rotations. Let $\mathcal{F}_i'$ be an intermediate reference frame which is achieved from $\mathcal{F}_i^0$ by rotating an angle θ_i about the $\hat{k}_i^0$ axis (positive according to the RHR). $\mathcal{F}_i$ is achieved from $\mathcal{F}_i'$ by a rotation of angle φ_i about the $\hat{j}_i'$ axis (positive according the RHR). Concatenating the respective transformations yields

$$\begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}_i = \begin{bmatrix} c\varphi\, c\theta & c\varphi\, s\theta & -s\varphi \\ -s\theta & c\theta & 0 \\ s\varphi\, c\theta & s\varphi\, s\theta & c\varphi \end{bmatrix}_i \begin{pmatrix} \hat{i}^0 \\ \hat{j}^0 \\ \hat{k}^0 \end{pmatrix}_i$$

or in terms of the notation of Hughes[3]

$$\mathcal{F}_i = [C_{i0}]\,\mathcal{F}_i^0$$

Above we have introduced the short-hand notation: $c\varphi = \cos\varphi$, $s\varphi = \sin\varphi$, and similarly for θ. The transformation from $\mathcal{F}_i$ to $\mathcal{F}_R$ is expressed as

$$\mathcal{F}_R = [C_{Ri}]\,\mathcal{F}_i$$

where

$$[C_{Ri}] = [C_{Ri}^0]\,[C_{i0}]^T$$

Let $\ell_i(t)$ be the straight-line distance between points A_i and P_i. Then

$$\bar{p}_i = \ell_i(t)\,\hat{i}_i$$

The geometry is illustrated in Figure-3.

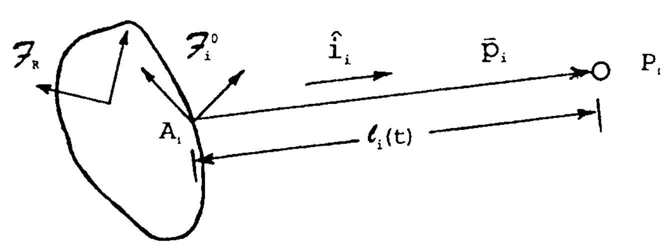

Figure-3. Tether Reference Frame Geometry.

Resolved into components referred to frame $\mathcal{F}_R$, we write

$$\vec{p}_i = \mathcal{F}_R^T \underline{p}_i$$

with

$$\underline{p}_i = [C_{Ri}^0] \begin{pmatrix} \ell c\varphi\, c\theta \\ \ell c\varphi\, s\theta \\ -\ell s\varphi \end{pmatrix}_i$$

The time derivative of $\vec{p}_i$ with respect to frame $\mathcal{F}_R$ is expressed as

$$\overset{\circ}{\vec{p}}_i = \mathcal{F}_R^T \dot{\underline{p}}_i$$

where

$$\dot{\underline{p}}_i = [C_{Ri}^0]\,[D_i]\,\dot{\underline{q}}_i$$

and where we have introduced

$$\underline{q}_i = \begin{pmatrix} \ell_i \\ \theta_i \\ \varphi_i \end{pmatrix}$$

and

$$[D_i] = \begin{bmatrix} c\varphi\, c\theta & -\ell c\varphi\, s\theta & -\ell s\varphi\, c\theta \\ c\varphi\, s\theta & \ell c\varphi\, c\theta & -\ell s\varphi\, s\theta \\ -s\varphi & 0 & -\ell c\varphi \end{bmatrix}_i$$

The time derivative of $[D_i]$ is recorded as

$$[\dot{D}_i] = \begin{bmatrix} -\left(\dot{\varphi}\, s\varphi\, c\theta + \dot{\theta}\, c\varphi\, s\theta\right) & -\dot{\ell} c\varphi\, s\theta + \ell\left(\dot{\varphi}\, s\varphi\, s\theta - \dot{\theta}\, c\varphi\, c\theta\right) & -\dot{\ell} s\varphi\, c\theta + \ell\left(\dot{\theta}\, s\varphi\, s\theta - \dot{\varphi}\, c\varphi\, c\theta\right) \\ \left(\dot{\theta}\, c\varphi\, c\theta - \dot{\varphi}\, s\varphi\, s\theta\right) & \dot{\ell} c\varphi\, c\theta - \ell\left(\dot{\varphi}\, s\varphi\, c\theta + \dot{\theta}\, c\varphi\, s\theta\right) & -\dot{\ell} s\varphi\, s\theta - \ell\left(\dot{\varphi}\, c\varphi\, s\theta + \dot{\theta}\, s\varphi\, c\theta\right) \\ -\dot{\varphi}\, c\varphi & 0 & -\dot{\ell} c\varphi + \ell\dot{\varphi}\, s\varphi \end{bmatrix}_i$$

The vector quantities; $\vec{a}_i$, $\vec{r}_c$, $\vec{\omega}$, $\vec{v}_0$, $\vec{F}_R$, $\vec{G}_R$, $\vec{L}_i$, $\vec{L}$, $\vec{H}_0$, $\vec{H}_c$ are expressed in scalar components resolved in $\mathcal{F}_R$ as the 3x1 column matrices; $\underline{a}_i$, $\underline{r}_c$, $\underline{\omega}$, $\underline{v}_0$, $\underline{F}_R$, $\underline{G}_R$, $\underline{L}_i$, $\underline{L}$, $\underline{H}_0$, $\underline{H}_c$. Similarly, the dyadics; $\vec{I}_R$ and $\vec{J}_i$ are expressed in scalar form resolved in $\mathcal{F}_R$ as the 3x3 matrices; $[I_R]$, $[J_i]$. The force vectors; $\vec{F}_i$ and $\vec{T}_i$ are resolved

into components referred to $\mathcal{F}_i$, and expressed as the 3x1 column matrices $\underline{F}_i$ and $\underline{T}_i$.

Resolved into components referred to $\mathcal{F}_R$, Eq.(2) can be written as

$$\underline{L}_i = m_i \underline{v}_0 - m_i(\underline{a}_i + \underline{p}_i)^{\times}\underline{\omega} + [E_i]\,\dot{\underline{q}}_i$$

where we have introduced

$$[E_i] = m_i[C^0_{Ri}]\,[D_i]$$

and the skew symmetric matrix cross product operator notation:

$$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad \Rightarrow \quad \underline{b}^{\times} = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$$

Equation (7) resolved in $\mathcal{F}_R$ is written as

$$m_i \dot{\underline{v}}_0 - m_i(\underline{a}_i + \underline{p}_i)^{\times}\dot{\underline{\omega}} + [E_i]\,\ddot{\underline{q}}_i = [C_{Ri}]\,(\underline{F}_i + \underline{T}_i) + \underline{N}_i \tag{10}$$

where

$$\underline{N}_i = -\underline{\omega}^{\times}\,(\underline{L}_i + [E_i]\,\dot{\underline{q}}_i) - [\dot{E}_i]\,\dot{\underline{q}}_i$$

and

$$[\dot{E}_i] = m_i[C^0_{Ri}]\,[\dot{D}_i]$$

The system linear momentum, Eq.(3) resolved in $\mathcal{F}_R$, is written as

$$\underline{L} = m_t \underline{v}_0 - m_t \underline{r}_c^{\times}\underline{\omega} + \sum_{i=1}^{N} [E_i]\,\dot{\underline{q}}_i \tag{11}$$

The position of the system centroid with respect to point-0 is

$$\underline{r}_c = \frac{1}{m_t}\sum_{i=1}^{N} m_i(\underline{a}_i + \underline{p}_i)$$

In scalar form Eq.(8) is expressed as

$$m_t \dot{\underline{v}}_0 - m_t \underline{r}_c^{\times}\dot{\underline{\omega}} + \sum_{i=1}^{N} [E_i]\,\ddot{\underline{q}}_i = \underline{F}_R + \sum_{i=1}^{N} [C_{Ri}]\,\underline{F}_i + \underline{N}_L \tag{12}$$

where

$$\underline{N}_L = -\underline{\omega}^{\times}\,(\underline{L} + \sum_{i=1}^{N} [E_i]\,\dot{\underline{q}}_i\,) - \sum_{i=1}^{N} [\dot{E}_i]\,\dot{\underline{q}}_i$$

The centroidal angular momentum of the system, from Eq.(6) becomes

$$\underline{H}_c = \left\{[I_R] + \sum_{i=1}^{N}[J_i] - m_t(\underline{r}_c^T\underline{r}_c[1] - \underline{r}_c\underline{r}_c^T)\right\}\underline{\omega} + \sum_{i=1}^{N}(\underline{a}_i + \underline{p}_i - \underline{r}_c)^{\times}[E_i]\,\underline{\dot{q}}_i \quad (13)$$

where [1] is the 3x3 identity matrix. The scalar form of Eq.(5), and its time derivative, are recorded as

$$[J_i] = m_i\left[(\underline{a}_i+\underline{p}_i)^T(\underline{a}_i+\underline{p}_i)[1] - (\underline{a}_i+\underline{p}_i)(\underline{a}_i+\underline{p}_i)^T\right]$$

$$[\dot{J}_i] = m_i\left[2(\underline{a}_i+\underline{p}_i)^T\underline{\dot{p}}_i[1] - (\underline{a}_i+\underline{p}_i)\underline{\dot{p}}_i^T - \underline{\dot{p}}_i(\underline{a}_i+\underline{p}_i)^T\right]$$

The system rotational equilibrium equation given by Eq.(9) is written as

$$m_t\underline{r}_c^{\times}\underline{\dot{v}}_0 + \left([I_R] + \sum_{i=1}^{N}[J_i]\right)\underline{\dot{\omega}} + \sum_{i=1}^{N}(\underline{a}_i+\underline{p}_i)^{\times}[E_i]\,\underline{\ddot{q}}_i = \underline{G}_R + \sum_{i=1}^{N}(\underline{a}_i+\underline{p}_i)^{\times}[C_{Ri}]\,\underline{F}_i + \underline{N}_H \quad (14)$$

where

$$\underline{N}_H = \left(m_t\underline{r}_c^{\times}\underline{v}_0^{\times} - \sum_{i=1}^{N}[\dot{J}_i]\right)\underline{\omega} - \underline{\omega}^{\times}\left[\left([I_R] + \sum_{i=1}^{N}[J_i]\right)\underline{\omega} + \sum_{i=1}^{N}(\underline{a}_i+\underline{p}_i)^{\times}[E_i]\,\underline{\dot{q}}_i\right] - \sum_{i=1}^{N}(\underline{a}_i+\underline{p}_i)^{\times}[\dot{E}_i]\,\underline{\dot{q}}_i$$

Equations (10), (12), and (14) constitute 6+3N independent scalar equilibrium equations for the system. The system motion equations assembled in matrix form, in the order; (12), (14), and (10), (i=1,2,...,N), (with the last premultiplied by $[D_i]^T[C_{Ri}^0]^T$), are

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}\begin{bmatrix} \underline{\dot{v}}_0 \\ \underline{\omega} \\ \underline{\ddot{q}} \end{bmatrix} = \begin{bmatrix} \underline{N}_L \\ \underline{N}_H \\ \underline{N}_q \end{bmatrix} + \begin{bmatrix} \underline{F}_L \\ \underline{F}_H \\ \underline{F}_q \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{T}_q \end{bmatrix} \quad (15)$$

The premultiplication of Eq.(10) by $[D_i]^T[C_{Ri}^0]^T$ was necessary to achieve a symmetric generalized mass matrix. The notation introduced in Eq.(15) is defined below.

$$\underline{q} = \begin{bmatrix} \underline{q}_1 \\ \underline{q}_2 \\ \vdots \\ \underline{q}_N \end{bmatrix}; \qquad \underline{N}_q = \begin{bmatrix} [D_1]^T[C^0_{R1}]^T\underline{N}_1 \\ [D_2]^T[C^0_{R2}]^T\underline{N}_2 \\ \vdots \\ [D_N]^T[C^0_{RN}]^T\underline{N}_N \end{bmatrix}$$

$$\underline{F}_q = \begin{bmatrix} [D_1]^T[C_{01}]\,\underline{F}_1 \\ [D_2]^T[C_{02}]\,\underline{F}_2 \\ \vdots \\ [D_N]^T[C_{0N}]\,\underline{F}_N \end{bmatrix}; \qquad \underline{T}_q = \begin{bmatrix} [D_1]^T[C_{01}]\,\underline{T}_1 \\ [D_2]^T[C_{02}]\,\underline{T}_2 \\ \vdots \\ [D_N]^T[C_{0N}]\,\underline{T}_N \end{bmatrix}$$

$$\underline{F}_L = \underline{F}_R + \sum_{i=1}^{N} [C_{Ri}]\,\underline{F}_i$$

$$\underline{F}_H = \underline{G}_R + \sum_{i=1}^{N} (\underline{a}_i + \underline{p}_i)^{\times}[C_{Ri}]\,\underline{F}_i$$

The partitions of the generalized mass matrix are:

$$M_{11} = m_t[\,1\,]$$

$$M_{12} = M_{21}^T = -m_t\underline{r}_c^{\times}$$

$$M_{13} = M_{31}^T = \big[[E_1]\ \ [E_2]\ \ \cdots\ \ [E_N]\big]$$

$$M_{22} = [I_R] + \sum_{i=1}^{N} [J_i]$$

$$M_{23} = M_{32}^T = \big[(\underline{a}_1 + \underline{p}_1)^{\times}[E_1] \,\vdots\, (\underline{a}_2 + \underline{p}_2)^{\times}[E_2] \,\vdots\, \cdots \,\vdots\, (\underline{a}_N + \underline{p}_N)^{\times}[E_N]\big]$$

$$M_{33} = \begin{bmatrix} m_1[D_1]^T[D_1] & [0] & \cdots & [0] \\ [0] & m_2[D_2]^T[D_2] & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots \\ [0] & [0] & \cdots & m_N[D_N]^T[D_N] \end{bmatrix}$$

where

$$m_i[D_i]^T[D_i] = \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i\ell_i^2 c^2\varphi_i & 0 \\ 0 & 0 & m_i\ell_i^2 \end{bmatrix}$$

The generalized mass matrix is positive definite, provided consideration is restricted to $\ell_i(t) > 0$ and $\varphi_i \neq \pm\, \pi/2$ (i=1,2,...,N).

Tether Tension

The tether force acting on P_i is $\bar{T}_i$. When in a tensile state, $\bar{T}_i$ is directed from P_i to A_i. From the definition of frame $\mathcal{F}_i$ we have

$$\underline{T}_i = -T_i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -T_i\, \underline{e}_1$$

where T_i is the magnitude of the tether tension. From the definitions of $[D_i]$ and $[C_{0i}]$, the generalized tension force term in Eq.(15) simplifies to

$$\underline{T}_q^T = \begin{pmatrix} -T_1 \underline{e}_1^T & -T_2 \underline{e}_1^T & \cdots & -T_N \underline{e}_1^T \end{pmatrix} \tag{16}$$

Extensible Tether

To accommodate an extensible tether model a constitutive law is required. Two such elementary relations are considered here. The subscript identifier i is dropped here for clarity. Let $\ell_0(t)$ denote the unstrained length of tether deployed at time t. The longitudinal strain of the tether, for our idealization, is defined as

$$\varepsilon = \frac{\ell(t) - \ell_0(t)}{\ell_0(t)}$$

The longitudinal stress in the tether is simply

$$\sigma = \frac{T}{A}$$

where A is the cross-sectional area of the load carrying material. For a linear elastic material the stress-strain relation is

$$\sigma = E\varepsilon$$

where E is the modulus of elasticity. Combining the above, an expression for the tension of a linear elastic tether material is obtained

$$T = \frac{EA}{\ell_0(t)} \left(\ell(t) - \ell_0(t) \right); \qquad \ell \geq \ell_0 \tag{17}$$

By admitting more general forms of the stress-strain relation more complicated material behaviors can be considered. Simple energy dissipation can be introduced by adding a stretch rate dependent term,

$$T = \frac{EA}{\ell_0(t)} (\ell(t) - \ell_0(t)) + C(\dot{\ell} - \dot{\ell}_0) ; \qquad \ell \geq \ell_0 \tag{18}$$

where C is a damping coefficient to be specified. Since the tether can not resist compression the tension force is taken to be identically zero for $\ell < \ell_0$. Equations (17) or (18) are sufficient to specify the tension, provided $\ell_0(t)$ is known, as for example would be the case with a motorized deployer.

Inextensible Tether

For an inextensible tether there is no constitutive law. One can still proceed provided that either the tension is known in terms of a definite function of the system state variables and/or time, or if kinematic constraints can be specified. The former case can occur during deployments or retractions under the action of a mechanized controller, or during deployments in the presence of known frictional forces. Kinematic constraints arise when a tether is presumed to be taught and its length time history is known, a situation which is treated in the final section of this report.

Kinematics of R

The orientation of the body fixed frame $\mathcal{F}_R$ with respect to the inertial frame $\mathcal{F}_I$ is described in terms of a 1-2-3 Euler angle sequence. The orthogonal unit basis vectors $\hat{i}, \hat{j}, \hat{k}$ for each frame referred to below are denoted with appropriate subscript and superscript identifiers. The signs of all single axis rotations are positive according to the right hand rule. Let $\mathcal{F}_I^{'}$ be an intermediate frame achieved from $\mathcal{F}_I$ by a rotation of angle ψ_1 about the $\hat{i}_I$ axis. Let frame $\mathcal{F}_R^{'}$ be achieved from $\mathcal{F}_I^{'}$ by rotation of angle ψ_2 about the axis $\hat{j}_I^{'}$. $\mathcal{F}_R$ is achieved from $\mathcal{F}_R^{'}$ by a rotation of angle ψ_3 about the $\hat{k}_R$ axis. The direction cosine matrix transforming vector components from $\mathcal{F}_R$ to $\mathcal{F}_I$ is obtained by concatenating the sequence of rotation transformations and is denoted as:

$$[C_{IR}] = \begin{bmatrix} c\psi_2 c\psi_3 & -c\psi_2 s\psi_3 & s\psi_2 \\ c\psi_1 s\psi_3 + s\psi_1 s\psi_2 c\psi_3 & c\psi_1 c\psi_3 - s\psi_1 s\psi_2 s\psi_3 & -s\psi_1 c\psi_2 \\ s\psi_1 s\psi_3 - c\psi_1 s\psi_2 c\psi_3 & s\psi_1 c\psi_3 + c\psi_1 s\psi_2 s\psi_3 & c\psi_1 c\psi_2 \end{bmatrix}$$

where we have again used the short-hand notation: $c\psi = \cos\psi$, $s\psi = \sin\psi$.

The angular velocity of $\mathcal{F}_R$ with respect to $\mathcal{F}_I$ is

$$\vec{\omega} = \dot{\psi}_1\, \hat{i}_I + \dot{\psi}_2\, \hat{j}'_I + \dot{\psi}_3\, \hat{k}_R$$

Resolved into components referred to $\mathcal{F}_R$ we have

$$\underline{\omega} = \begin{bmatrix} c\psi_2 c\psi_3 & s\psi_3 & 0 \\ -c\psi_2 s\psi_3 & c\psi_3 & 0 \\ s\psi_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{bmatrix}$$

$$= [\Pi]\, \underline{\dot{\psi}} \tag{19}$$

Inverting the above equation yields

$$\begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{bmatrix} = \begin{bmatrix} c\psi_3/c\psi_2 & -s\psi_3/c\psi_2 & 0 \\ s\psi_3 & c\psi_3 & 0 \\ -c\psi_3 t\psi_2 & s\psi_3 t\psi_2 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

or more compactly,

$$\underline{\dot{\psi}} = [\Pi]^{-1}\, \underline{\omega} \tag{20}$$

where $t\psi_2 = \tan\psi_2$, and the singularity at $\psi_2 = \pm\pi/2$ is noted.

Recall that the position of point-0 with respect to point-I was given by $\vec{R}_0$. The time derivative of $\vec{R}_0$ observed in $\mathcal{F}_I$ is $\vec{v}_0$. The components of $\vec{v}_0$ resolved in $\mathcal{F}_R$ are $\underline{v}_0$. If we let $\underline{R}_0$ be the 3x1 column matrix of components of $\vec{R}_0$ referred to $\mathcal{F}_I$ then

$$\underline{\dot{R}}_0 = [C_{IR}]\, \underline{v}_0 \tag{21}$$

<u>System State Vector</u>

The instantaneous configuration of the system is given by the vector of generalized coordinates:

$$\underline{X}^T = \left(\underline{R}_0^T \;\; \underline{\psi}^T \;\; \underline{q}_1^T \;\; \underline{q}_2^T \;\; \cdots \;\; \underline{q}_N^T \right)$$

The motion equations for the system, Eq.(15), have been written in terms of the vector of velocities

$$\underline{V}^T = \left(\underline{v}_0^T \;\; \underline{\omega}^T \;\; \underline{\dot{q}}_1^T \;\; \underline{\dot{q}}_2^T \;\; \cdots \;\; \underline{\dot{q}}_N^T \right)$$

The mechanical state of the system is thus specified entirely by the vector

$$\{Y\} = \left\{ \begin{array}{c} \underline{V} \\ \cdots \\ \underline{X} \end{array} \right\} \tag{22}$$

and its time derivative, given by

$$\{\dot{Y}\} = \left\{ \begin{array}{c} [M]^{-1}\left(\underline{N} + \underline{F} + \underline{T} \right) \\ \cdots\cdots\cdots\cdots \\ [C_{IR}]\, \underline{v}_0 \\ [\Pi]^{-1}\, \underline{\omega} \\ \underline{\dot{q}}_1 \\ \vdots \\ \underline{\dot{q}}_N \end{array} \right\} \tag{23}$$

where the upper partition of the RHS corresponds to Eq.(15).

<u>Prescribed Motion</u>

In the development of control systems for complicated mechanical systems it is often helpful to be able to determine the forces and torques which must act to produce the desired motions. In this section the foregoing formulation is modified to allow for the specification of any or all the degrees of freedom. A specified or prescribed degree of freedom is assumed to have a known time history, i.e. the coordinate and its time derivatives are known a priori.

The dynamic equilibrium equations, Eq.(15), are here recast in terms of the time derivatives of the coordinates of $\underline{X}$ rather than those of $\underline{V}$. Equations (20) and (21) provide the transformations between the nonholonomic velocities $\underline{v}_0$ and $\underline{\omega}$ and the time derivatives of the position and attitude coordinates $\underline{R}_0$ and $\underline{\psi}$. Thus

$$\underline{V} = [B]\,\dot{\underline{X}}$$

where

$$[B] = \begin{bmatrix} [C_{RI}] & [0] & [0] \\ [0] & [\Pi] & [0] \\ [0] & [0] & [1] \end{bmatrix}$$

It follows that

$$\dot{\underline{V}} = [B]\,\ddot{\underline{X}} + \underline{N}_X \tag{24}$$

where

$$\underline{N}_X = \begin{bmatrix} [\dot{C}_{RI}]\,\dot{\underline{R}}_0 \\ [\dot{\Pi}]\,\dot{\underline{\psi}} \\ \underline{0} \end{bmatrix}$$

$$[\dot{C}_{RI}] = -\underline{\omega}^{\times}\,[C_{RI}]$$

$$[\dot{\Pi}] = \left[\begin{array}{c:c:c} -(\dot{\psi}_2 s\psi_2 c\psi_3 + \dot{\psi}_3 c\psi_2 s\psi_3) & \dot{\psi}_3 c\psi_3 & 0 \\ \hdashline \dot{\psi}_2 s\psi_2 s\psi_3 - \dot{\psi}_3 c\psi_2 c\psi_3 & -\dot{\psi}_3 s\psi_3 & 0 \\ \hdashline \dot{\psi}_2 c\psi_2 & 0 & 0 \end{array}\right]$$

Substituting Eq.(24) into Eq.(15), we write

$$[M][B]\,\ddot{\underline{X}} = (\,\underline{N} - [M]\,\underline{N}_X\,) + \underline{F} + \underline{T} \tag{25}$$

The coefficient matrix of the accelerations can be made symmetric by premultiplication of Eq.(25) by $[B]^T$,

$$[B]^T\,[M][B]\,\ddot{\underline{X}} = [B]^T\,(\,\underline{N} - [M]\,\underline{N}_X) + [B]^T\,\underline{F} + \underline{T}$$

or

$$[M']\,\ddot{\underline{X}} = \underline{N}' + \underline{F}' + \underline{T} \tag{26}$$

where the primes denote the transformed quantities, and it is noted that $[B]^T\underline{T} = \underline{T}$.

For prescribed motion in any n of the coordinates contained in $\underline{X}$ the constraint equations assume the particularly simple form

$$[S_p]^T\underline{X} = \underline{x}_p(t) \tag{27}$$

where $[S_p]$ is a simple constant boolean selection matrix of dimension $(6+3N)$xn, and $\underline{x}_p(t)$ are n specified functions of time. The time derivative of Eq.(27) is

$$[S_p]^T \dot{\underline{X}} = \dot{\underline{x}}_p(t) \tag{28}$$

The equations of motion given by Eq.(26) correspond to the unconstrained system. If certain of the degrees of freedom are to be specified it is evident that some set of constraint forces must act to enforce the designated motion. For the constraints given by Eq.(28) it is known from analytical mechanics[4] that the right hand side of Eq.(26) must be augmented by generalized constraint forces of the form $[S_p]\underline{\lambda}$, where $\underline{\lambda}$ is an nx1 column matrix of Lagrange multipliers. The augmented form of the motion equations subject to the aforementioned constraints are

$$[M']\ddot{\underline{X}} = \underline{N}' + \underline{F}' + \underline{T} + [S_p]\underline{\lambda} \tag{29}$$

Differentiating Eq.(28) with respect to time then allows the constraints and motion equations to be written together as

$$\begin{bmatrix} [M'] & -[S_p] \\ -[S_p]^T & \underline{0}^T \end{bmatrix} \begin{pmatrix} \ddot{\underline{X}} \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{N}' + \underline{F}' + \underline{T} \\ -\ddot{\underline{x}}_p(t) \end{pmatrix} \tag{30}$$

One of a number of solution procedures is to solve Eq.(30) simultaneously for $\ddot{\underline{X}}$ and $\underline{\lambda}$, then numerically integrate the system state equations.

Physical Forces of Constraint

To determine the scalar components of the physical constraint forces and torques required to enforce the prescribed motion the correspondence between those elements and the generalized forces of constraint, given by $[S_p]\underline{\lambda}$, must be established. Referring to Eq.(15), and the immediately following definitions, the vector of generalized forces $\underline{F}$ can be written in terms of the resultants of the external applied forces and torques acting on the system components as

$$
\underline{F} = \begin{bmatrix} \underline{F}_L \\ \underline{F}_H \\ \underline{F}_q \end{bmatrix} = \left[\begin{array}{c:c:cccc} [1] & [0] & [C_{R1}] & [C_{R2}] & \cdots \\ \hdashline [0] & [1] & (\underline{a}_1+\underline{p}_1)^{\times}[C_{R1}] & (\underline{a}_2+\underline{p}_2)^{\times}[C_{R2}] & \cdots \\ \hdashline [0] & [0] & [D_1]^T[C_{01}] & [0] & \cdots \\ [0] & [0] & [0] & [D_2]^T[C_{02}] & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right] \begin{bmatrix} \underline{F}_R \\ \underline{G}_R \\ \underline{F}_1 \\ \underline{F}_2 \\ \vdots \end{bmatrix}
$$

or

$$
\underline{F} = [A]\,\underline{F}_a
$$

From Eq.(29) we observe that the external applied forces $\underline{F}_a$ are related to the corresponding generalized forces by

$$
\underline{F}' = [B]^T [A]\,\underline{F}_a
$$

Any constraint forces which act in addition to the external applied forces must "connect" to the acceleration variables in Eq.(29) in a manner analogous to that of $\underline{F}_a$. Let $\underline{F}_c$ be the constraint forces enforcing the prescribed motion which act in parallel with those of $\underline{F}_a$. Then the components of the physical constraint forces are determined from the Lagrange multipliers by

$$
\underline{F}_c = ([B]^T [A])^{-1} [S_p]\,\underline{\lambda} \tag{31}
$$

References

[1] Quinn, D. A. and D. C. Folta, "A Tethered Formation Flying Concept for the SPECS Mission", Paper AAS 00-015, 23rd Annual AAS Guidance and Control Conference, Feb. 2-6, 2000, Breckenridge, CO.

[2] Wittenburg, J., "Dynamics of Systems of Rigid Bodies", B. G. Teubner, Stuttgart, 1977, p34 & p39.

[3] Hughes, P. C., "Spacecraft Attitude Dynamics", J. Wiley & Sons, 1986, Appendix B.

[4] Shabana, A. A., "Dynamics of Multibody Systems", 2nd Edition, Cambridge University Press, 1998, p247.